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COMBINED RESIDENTIAL LOCATION AND TRANSPORTATION MODELS

by

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ABSTRACT

This paper presents predictive models which combine residential location with transportation. These models predict residential location and the use and performance of the transportation system for exogenously given total housing stock and employment locations. One can use them to determine the impact on the transportation system of changes in the supply of housing or reciprocally to determine the impact on housing choices of new investments in the transportation system. They are formulated as mathematical programs generalizing the Herbert-Stevens model of residential location to include previously proposed equilibrium models of trip assignment, trip distribution and modal choice. A solution methodology is proposed for prediction and calibration.

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I Models of residential location and transportation

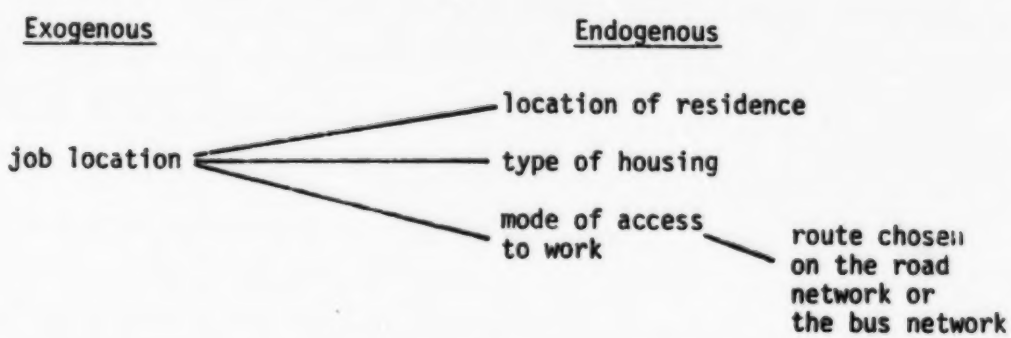
A Modelling transportation and residential location choices

Many practical urban planning problems require the prediction of the impact of transportation decisions (new investments, changes in transit fares, etc...) on residential location, and also the prediction of the impact of changes in the supply of housing on the use and performance of the transportation system. These impacts have to be predicted at a sufficient level of spatial detail to be of practical use to urban and transportation planners. These planning problems require the development of combined residential location - transportation models. It is the purpose of this paper to present such models and to explore their solution methods.

More precisely this paper will address the problem of predicting for the medium-term (about 5 years) the demand for housing, the demand for different modes of travel to work, the actual utilization and level of performance of the transportation system. A suitable disaggregation of households, at least by income group , will be assumed,as well as a suitable disaggregation of housing by type and location. The spatial distribution and nature of the housing stock, as well as the transportation system (road network, bus network, bus frequency, etc...) will be exogenous to the model. The price of housing and the money cost of travel by different modes will be assumed known. The total number of households as well as their distribution among different types will be given exogenously. Finally the total amount of employment and its spatial distribution will also be exogenous to the model. The family of models

presented will focus on trips to work and will neglect other trip purposes. The dimensions of choice left to each household are described in figure 1. For a given household the choice of residence is assumed to be a tradeoff between accessibility to work and the characteristics of the zone and dwelling chosen, for the type of household to which this particular

Figure 1: Choices open to each household



household belongs, characterized by its income and its tastes The need for joint models of residential location and transportation stems from the interdependencies that exist between the choices open to households. Assuming a city where the bus service is excellent in the center but bad at the periphery, there will be a tendency for a household to choose either the (access to work by bus, central city residence) bundle or the (access to work by car, suburban residence) bundle. Also the "accessibility" advantage of a residential location depends on the performance of the transportation system (e.g. congestion on the road network) which itself depends on modal choice and route choice. The assumption of exogenous supply of housing and transportation infrastructure is made because of the *durability* of physical infrastructure. Even if, in the very long run, supply decisions

on housing or transportation infrastructure might, in principle, adjust to shifts in demand, in the short or medium term (1 to 5 years) this is only partially the case and it is of interest to try to predict possible imbalances either in the housing market, expressing themselves for instance in vacancies for some types of housing in some locations, or in the transportation system, expressing themselves in excessive congestion on some links of the road network or underutilization of the transit system.

B The state of the art

Three different approaches have been proposed to date for modelling the interdependencies between residential location and transportation: general equilibrium simulation (Arnott and MacKinnon, 1976), disaggregate demand models (Lerman, 1976), zonal aggregate models (Putman, 1975; Berechman, 1976, 1978; Boyce, 1976). We will examine in turn the strengths and weaknesses of these previous approaches.

1) The general equilibrium approach

Arnott and MacKinnon (1976) proposed a general equilibrium simulation of the supply and demand for housing in a circular city where all employment is located at the center and where the commuters have the choice between several modes of access to work (walking, public transportation and the automobile) with different characteristics. The price of land and housing is determined endogenously while transportation decisions (such as fares) are exogenous. The equilibrium is obtained by means of a fixed point algorithm.

This model is theoretically sound and very elegant. The major

advantages of this approach are that it allows a very precise study of the welfare impacts of different transportation measures and that it treats housing supply and housing prices endogenously. On the other hand, housing is treated as a non-durable commodity which makes the model more suitable for very long run studies. The spatial representation of the transportation system is highly simplified and there is no modelling of automobile congestion. Only two groups of households are considered and only one point of employment is assumed. These limitations make the model unsuitable for practical use in urban and transportation planning, where detailed predictions are required for the short or medium term. As recognized by the authors themselves the reason preventing a more realistic representation of the problem is computational rather than theoretical. Thus, given the present state of the art of fixed point algorithms, only qualitative studies at a very aggregate level can be done by means of a general equilibrium simulation.

2) The disaggregate demand modelling approach

Lerman (1976) has used a multinomial logit model at a disaggregated level to predict the joint choice of residence and mode of access to work. This approach has been integrated into the N.B.E.R. Urban simulation package (Ingram *et al.*, 1972). The major advantages of this approach are the ease and relatively low cost of calibration and prediction, and the detailed behavioral level at which choices can be explained: many individual characteristics such as age, ethnic group, income, sex, etc. can be taken into account. The main theoretical

limitation of the multinomial logit model (the so-called attribute-correlation problem) has recently been partially overcome by McFadden (1978) and Williams (1977) so that the disaggregate demand modelling approach seems very promising indeed.

Unfortunately this approach focuses exclusively on the *demand* side for housing or transportation. Therefore this methodology can be extremely good for *explaining demand behavior* and it can do that in a very refined way. However it is unsuitable for *prediction* because of the existence of supply constraints that make themselves felt only in the *aggregate*. Examples of such supply constraints are the limited parking capacity of some areas, which limits work-trips by auto to those areas, the limited supply of housing which constrains residential choices and the limited capacity of the road network which results in automobile congestion. Demand predictions with the use of disaggregate models can exceed the available capacities of housing or parking and understate the impact of automobile congestion on residential choice and modal choice: there is no automatic feedback of aggregate supply constraints on individual behavior. This can be done in an *ad hoc* fashion by introducing artificial shadow prices until the constraints are met: for instance Lerman (1976) used iterations on house prices to adjust demand for housing to the existing supply in integrating his model into the N.B.E.R. urban simulation package. This procedure presents two problems. Firstly, it is heuristic and multiple predictions can be obtained in this fashion: there may be a large number of ways of meeting the constraints and there is a priori no way of knowing

which one to choose. The problem of multiple solutions is compounded when several supply constraints exist simultaneously (for instance congestion on the road network and limited supply of the housing stock). Policy-sensitivity of the model requires the uniqueness of the solution predicted. Eventual multiple solutions should result from the nature of the system being modelled rather than from the heuristic nature of the prediction procedure. Secondly, in reality, there may be rigidity in some of the prices (for instance if there is complete rent control or if there is a slow adjustment in parking prices). In that case it is unsound to meet the supply constraints by manipulating the prices. The model should be able to predict vacancies and binding constraints. This in itself is a valuable information to urban planners: vacancies are indeed observed. A relevant predictive model should be able to provide guidance in the determination of the type and location of new housing to be produced by either the public sector or the private sector. The dual variables associated with the supply constraints of a mathematical programming approach would provide such an information (See Hårsman and Snickars, 1975). This cannot be done with a disaggregate demand approach.

3) Aggregate residential location and transportation models

There is now a long tradition of aggregate residential location modelling[1]. Three kinds of aggregate modelling approaches have been proposed. First the Lowry model (1964) and different maximum-entropy formulations of the Lowry model (Wilson, 1970; Senior and Wilson, 1973) in which residential location is essentially explained by accessibility

1. See Senior (1973, 1974) for an excellent review of the state of the art until 1974.

to work. Second, linear programming models based on microeconomic theory in which preferences for different types of housing are introduced in addition to accessibility to work, in order to explain residential location. The different versions of the Herbert-Stevens model are well-known examples of such linear programming approaches (See Herbert and Stevens, 1960; Harris, 1962, 1966; Harris and Wheaton, 1972; Hårsman and Snickars, 1975; Wheaton, 1974). Third, syntheses of these two approaches have been proposed by Anas (1973, 1975) and Senior and Wilson (1974). Recently Williams and Senior (1978) have reformulated and reinterpreted the Senior-Wilson residential location model. This third kind of model seems promising because it offers a compromise between operationality and theoretical soundness. An important reason justifying the use of mathematical programming approaches over alternative approaches is the interpretation of the dual variables (See Hårsman and Snickars, 1975, for instance).

In parallel to the development of residential location models, transportation modelling was making great progress, both from a theoretical point of view and from a computational point of view. Precise definitions of traffic equilibrium were given and used as a basis for efficient mathematical programming techniques (see Florian, 1976; Florian and Nguyen, 1975). It was shown that trip distribution and traffic assignment could be solved simultaneously (see Evans, 1976; Florian, Nguyen and Ferland, 1975) and that modal choice could be introduced as well (see Florian and Nguyen, 1978).

Until recently however, aggregate residential location models assumed that transport costs were known and constant and disregarded transportation

congestion. Reciprocally, transport models assumed that residential location, like employment location, was exogenously determined. Clearly,due to congestion on the road network, residential location and transportation behavior are interdependent: residential choice depends partly on accessibility to work, which depends on congestion in the transportation system for peak hour work trips, and congestion itself depends on peak hour travel demand which depends itself on residential location. This interdependence has to be modelled if one is not to understate the impact of transportation congestion on residential choices.

First attempts at combining a land use model (including in particular residential location) and a transportation model are due to Putman (1973, 1975) and Berechman (1976,1978). These authors proposed to use iteratively a land use model of the Lowry type and a traffic equilibrium technique: Putman (1973) uses an incremental loading traffic assignment technique while Berechman proposes to use a traffic equilibrium technique producing a "user equilibrium". The land-use model and the traffic equilibrium model are not *integrated* but *interfaced*: the travel costs resulting from the traffic assignment technique are used as input into the land use model. In turn the land use allocation resulting from the land use model determines a demand for travel between origin-destination pairs which is used as input into the traffic equilibrium technique, etc... Such an approach is not guaranteed to converge towards a meaningful equilibrium. It might converge towards different solutions depending on the starting solution.

What is required is a predictive method that produces a *unique* solution which has the following characteristics:

a) If the "equilibrium" travel times of that solution are used as input into the land use model, the latter should reproduce land use allocations which are identical to those of the solution.

b) If the demand for travel resulting from the land use model is used as input into the traffic equilibrium technique, the latter should reproduce the travel times of the solution.

These different points were raised by Boyce (1978) who showed that a combined Lowry-type residential location model and equilibrium traffic assignment had the same mathematical structure as the combined trip distribution-trip assignment problem for which Evans (1976) and Florian, Nguyen and Ferland (1975) have proposed two different algorithms which both are proved to converge to the correct equilibrium.

C Approach chosen in this paper

Ideally we would like to develop a predictive model

1) which is theoretically sound;
2) which allows some interaction between supply and demand for housing and transportation;
3) which is sensitive at a *detailed spatial level* to transportation policy measures such as bus fares, bus frequency, new investments in roads, etc... and to housing policy measures such as price changes, demolition or new housing construction;
4) which has some welfare interpretation;
5) which has sufficient behavioral content, i.e. which considers household preferences in as refined a way as possible;
6) which is computationaly feasible.

As the reader might have guessed, such an ideal is unreachable and a reasonable compromise has to be found for our purpose, which is to predict the housing and transportation choices of households in the medium term (in 5 years). A general equilibrium simulation meets criteria 1), 2), 4) but fails on criteria 3), 5) and 6). A disaggregate demand modelling approach succeeds on criteria 1), 3) 5) and 6), but fails on the others. On the other hand, some of the best previously mentioned aggregate residential location models and transportation models can potentially meet all the criteria sufficiently well for our purpose but need to be *integrated* in a consistent mathematical programming framework so that the interdependencies between residential location and transportation are explicitly taken into account.

This integration is precisely what this paper will try to accomplish. This paper will synthesize the Senior-Wilson (1974) version of the Herbert-Stevens residential location model with the Florian and Nguyen (1978) transportation model in a unified mathematical programming framework. A unique equilibrium will be shown to exist and methods for solving the mathematical program in which the predictive model (or family of models) is embodied and for calibration will be presented. Such an approach cannot have the elegance of a general equilibrium simulation or the detailed behavioral content of a disaggregate demand modelling approach but appears to be a reasonable compromise between policy-sensitivity and computational feasibility on one hand, behavioral content and theoretical soundness on the other hand.

The rest of the paper is organized as follows. In section II, a possible

version of the Herbert-Stevens model of residential location is stated. In section III, the model of section II is modified under the assumption that there is one mode of access to work, the private car, and that there is congestion on the road network. Two models are proposed. Methods of calibration and prediction are stated for the second model, which is a maximum entropy model. Section IV extends the second model of section III to the case of two modes, the private car and the bus mode and determines endogenously modal choice in addition to residential location. The possibility of peak hour trips and non-peak hour trips is taken into consideration. Section V discusses first possible methods for estimating the bid rents that are exogenous to the models presented in the preceding sections, then the limitations of the models, finally suggests some extensions and areas for further research and concludes this paper.

II A version of the Herbert-Stevens model of residential location

A Model assumptions and notation

We will state a model that is based on the versions of the Herbert-Stevens model proposed by Senior and Wilson (1974) and by Hårsman and Snickars (1975). We borrow from Senior and Wilson (1974) the work-trip interpretation due to the disaggregation of households by employment location and from Hårsman and Snickars (1975) the assumption of exogenous housing rents. The model predicts residential location when employment location on one hand and the quantity and location of the housing stock on the other hand are given exogenously. The model uses Alonso's (1964) bid rent concept to translate household residential preferences. The allocation of households to dwellings is such that total *net* bids are maximized,

i.e. such that the aggregate willingness to pay for housing is maximized. This formulation reflects the competition of households for housing and reflects the fact that dwellings tend to be attributed to the "virtual" highest bidders, even if the bids are not transformed into *actual* rents. The economic rationale for this model has been well reviewed in the literature and will not be restated here[1] in more detail.

We assume known the bid rents of households for different types of housing in different zones. All households of a given type have the same level of utility, prespecified in advance and independent of housing choice. The assumption of an exogenously given level of utility can be justified on the ground that the city studied is part of a system of cities for which the level of utility should be uniform for each household type. If it were not, households would migrate from one city where utility is low to one where it is higher until utility is again uniform across cities. On the other hand if the city studied is a closed system, one can justify a prespecified level of utility by assuming that levels of "expected utility" are historically determined i.e. are relatively constant over time or change in a predictable way. If none of these two assumptions is valid, the level of utility of each household type should be adjusted endogenously. This point will be discussed in more detail in section V. The household bid rent is a decreasing function of the level of utility. The rents

1. See Alonso (1964)' Harris (1966), Harris and Wheaton (1972), Herbert and Stevens (1960), Wheaton (1974), Senior (1974), Harsman and Snickars (1975).

actually paid are supposed to be exogenously given. In reality, in many cities, there is only *some* degree of control over housing prices (public housing, rent control, ...) rather than *total* control. The assumption of fixed housing prices will be discussed in section V in more detail. We make also the assumptions that bids are *linearly* decreasing with the value of the time spent travelling between workplace and residence and that the value of time is the same for all household types. The two preceding assumptions are made to allow the synthesis of the Herbert-Stevens model with transportation models. Finally the location of the job is supposed to be *prior* to the choice of a residence. This assumption is plausible in the U.S.A. or Canada but not in many European cities, such as London or Paris, where it is more plausible to assume that residences are more stable over time than jobs due to the rigidities in the housing market and to the limited supply of housing.

T_{ij}^{kw} is the number of working heads of households living in a type k house in zone i earning annual income W in a job in zone j.

H_i^k is the number of houses (or dwellings) of type k in zone i.

E_j^w is the number of jobs yielding income W in zone j.

r_i^k is the *actual* annual rent paid for a type k house in zone i.

$$b_{ij}^{kw} = \beta_{ij}^{kw} - \theta c_{ij}$$

where b_{ij}^{kw} is the bid rent by a (j,w) household for a type k house in zone i, c_{ij} is the time spent travelling between i and j, θc_{ij} is the value of that time and β_{ij}^{kw} is the bid rent by a (j,w) household for a type k house in zone i that would occur if travel time was zero. The functional form chosen for the bid rents will be explained and justified in more detail in

Section V below. We will assume temporarily that c_{ij} is fixed.

B The model

T_{ij}^{kw} is required to be a solution to the following problem.

$$(H.S.)\ \text{Max} \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw}\ \hat{b}_{ij}^{kw} \tag{1}$$

$$\text{s.t.} \quad \sum_i \sum_k T_{ij}^{kw} = E_j^w \qquad \text{all } j,w \tag{2}$$

$$\sum_j \sum_w T_{ij}^{kw} \leq H_i^k \qquad \text{all } i,k \tag{3}$$

$$T_{ij}^{kw} \geq 0 \tag{4}$$

$$\text{where} \quad \hat{b}_{ij}^{kw} = b_{ij}^{kw} - r_i^k \tag{5}$$

is the individual net bid.

(H.S.) is a transportation problem of linear programming in which (1) is the aggregate willingness to pay, (2) states that every household must find a dwelling and (3) states that the supply of the housing stock in a given zone constrains the number of households that can be allocated there. The dual problem is:

$$(H.S.D.):\ \text{Min} \sum_i \sum_k \alpha_i^k\ H_i^k + \sum_j \sum_w \nu_j^w\ E_j^w \tag{6}$$

$$\text{s.t.} \quad \alpha_i^k + \nu_j^w \geq \hat{b}_{ij}^{kw} \qquad \text{all } i,\ j,\ k,\ w \tag{7}$$

$$\alpha_i^k \geq 0 \tag{8}$$

$$\nu_j^w \text{ either sign}$$

α_i^k can be interpreted as a location rent, ν_j^w as a tax (if > 0) or subsidy (if < 0) necessary for every household to be located somewhere. The rule of complementary slackness entails:

$$\alpha_i^k + \nu_j^w \geq b_{ij}^{kw} \quad \text{for} \quad T_{ij}^{kw} = 0 \tag{9}$$

$$T_{ij}^{kw} > 0 \Rightarrow \alpha_i^k + \nu_j^w = b_{ij}^{kw} \tag{10}$$

$$\text{i.e.} T_{ij}^{kw} > 0 \Rightarrow \alpha_i^k + r_i^k = b_{ij}^{kw} - \nu_j^w \tag{11}$$

$$\alpha_i^k > 0 \Rightarrow \sum_j \sum_w T_{ij}^{kw} = H_i^k \tag{12}$$

$$H_i^k > \sum_j \sum_w T_{ij}^{kw} \Rightarrow \alpha_i^k = 0 \tag{13}$$

Equation (11) states that the household's ability to pay after tax and subsidy corrections is equal to the total price of the housing bundle to which it is allocated, even though the ν_j^w and α_i^k may remain implicit. Equation (11) can be restated as:

$$T_{ij}^{kw} > 0 \Rightarrow \alpha_i^k + r_i^k = \beta_{ij}^{kw} - \theta c_{ij} - \nu_j^w \tag{11'}$$

C Limitations of the present model

First, no account is taken of the possible dispersion of preferences among households of a given type. The linear programming formulation can generate an allocation of households to dwellings where some types of households can be completely absent from some zones or some housing types. This is too extreme a representation of actual household behavior. An entropy-maximizing approach can produce an allocation of households to dwellings that is not "all or nothing", and therefore is likely to be better calibrated against actual behavior (see Anas 1973, 1975; Senior and Wilson, 1974). A second limitation of (H.S.) for our purposes is the absence of modelling of transportation behavior. In reality, due to transportation

congestion on the road network, the time spent travelling between zone i and zone j , c_{ij}, is not fixed but dependent on the actual utilization of the road network. The demand for housing in a certain zone may be such that transportation congestion will be generated due to work-trips between that residential zone and zones of employment. If this is not taken into account in the model, the "accessibility" advantage of that zone will be overstated.

The models in the next section attempt to relax these two limiting assumptions.

III Models of residential location taking into account transportation congestion on the automobile network

A The Herbert-Stevens model with transportation congestion

We relax the assumption that the travel times, c_{ij}, are fixed. We assume that there is only one mode of transport, that every worker owns a car, that there is congestion on the road network and that there is a uniform car-occupancy factor so that it is indifferent to measure flows on the road network in vehicles or in persons.

A is the set of arcs a of the road network.

$s_a(v_a)$ is the congestion function on link a which we assume to be a convex and increasing function of the flow v_a ,

$h_{m,ij}$ is the path flow on path m between zones i and j .

We want to extend the Herbert-Stevens model so as to satisfy (11') with c_{ij} replaced by u_{ij} , the "user-equilibrium" travel cost between i and j .

More precisely we want a solution to the extended problem to satisfy the following relationships:

$$T_{ij}^{kw} > 0 \Rightarrow \alpha_i^k + \nu_j^w = \hat{\beta}_{ij}^{kw} - \theta u_{ij} \tag{14}$$

$$\alpha_i^k + \nu_j^w \geq \hat{\beta}_{ij}^{kw} - \theta u_{ij} \quad \text{for} \quad T_{ij}^{kw} = 0 \tag{15}$$

$$\text{with} \qquad \hat{\beta}_{ij}^{kw} = \beta_{ij}^{kw} - r_i^k \tag{16}$$

$$h_{m,ij} > 0 \Rightarrow \sum_{a \in A} s_a(v_a)\, \delta_{am,ij} = u_{ij} \tag{17}$$

$$h_{m,ij} = 0 \Rightarrow \sum_{a \in A} s_a(v_a)\, \delta_{am,ij} \geq u_{ij} \tag{18}$$

$$\text{with} \quad \delta_{am,ij} = \begin{cases} 1 & \text{if a belongs to path m between i and j} \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha_i^k > 0 \Rightarrow \sum_j \sum_w T_{ij}^{kw} = H_i^k \tag{19}$$

$$H_i^k > \sum_j \sum_w T_{ij}^{kw} \Rightarrow \alpha_i^k = 0 \tag{20}$$

(17) and (18) represent Wardrop's "user-equilibrium" conditions. They state that, at equilibrium on the road network, the routes chosen by the travellers are such that the travel times between each origin and each destination are the same for all routes chosen and less than on any other route (see Florian, 1976, for a thorough presentation of traffic equilibrium concepts and techniques and Florian and Nguyen, 1975 for an empirical validation of the user-equilibrium concept).

The required model is the following. We want T_{ij}^{kw} and $h_{m,ij}$ to be a possible solution to:

$$\text{(H.S.C.): Max} \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw} \tilde{B}_{ij}^{kw} - \theta \sum_{a \in A} \int_0^{v_a} s_a(x)dx \qquad (21)$$

$$\text{s.t.} \sum_k \sum_w T_{ij}^{kw} = \sum_m h_{m,ij} \qquad \text{all } i,j \qquad (22)$$

$$\sum_i \sum_k T_{ij}^{kw} = E_j^w \qquad \text{all } j,w \qquad (23)$$

$$\sum_j \sum_w T_{ij}^{kw} \leq H_i^k \qquad \text{all } i,k \qquad (24)$$

$$v_a = \sum_m \sum_i \sum_j \delta_{am,ij} h_{m,ij} \qquad \text{all } a$$

$$T_{ij}^{kw} \geq 0 \qquad (25)$$

$$h_{m,ij} \geq 0 \qquad (26)$$

(H.S.C.) is the maximization of a concave function under linear constraints. The Kuhn-Tucker conditions are necessary and sufficient for optimality.

Let α_i^k be the dual variable associated with constraint (24), ν_j^w be the dual variable associated with constraint (23) and $U_{ij} = \theta u_{ij}$ be the dual variable associated with the conservation of flows constraint (22). It can be shown that the Kuhn-Tucker conditions of (H.S.C.) are the conditions (14), (15), (17), (18), (19) and (20). (H.S.C.) is therefore the required generalization of the Herbert-Stevens model (H.S.), when there is congestion on the transportation network, and only one mode, the private automobile.

B A maximum-entropy formulation of the Herbert-Stevens residential location model with transportation congestion on the automobile network

Entropy maximization methodology can be justified on different grounds:

1) On information-theoretical grounds (see Wilson, 1970).
2) On the ground that it accounts for the possibility of some degree of suboptimality in the system being modelled, here the housing market (Evans, 1973 and Wilson and Senior, 1974).

3) On the ground that a relationship can be established between the objective function of the maximum entropy formulation of an aggregate demand model and the Hotelling measure of consumer surplus (see Champernowne, Williams and Coelho, 1976).

4) On the ground that the objective function of a maximum-entropy formulation of a spatial interaction model can be interpreted as a "group locational surplus" directly on the basis of random utility theory. In this interpretation the entropy part of the objective function accounts for the dispersion of preferences within one type of households (see Williams and Senior, 1978).

We require T_{ij}^{kw} and $h_{m,ij}$ to be the solution to the following minimization problem:

$$(P1)\ \text{Min} \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw} \ell n\, T_{ij}^{kw} - \mu \{\sum_w \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw} T_{ij}^{kw} - \theta \sum_{a \in A} \int_0^{v_a} s_a(x)dx\} \tag{27}$$

s.t.

$$\sum_i \sum_k T_{ij}^{kw} = E_j^w \qquad \text{all } j,w \tag{28}$$

$$\sum_j \sum_w T_{ij}^{kw} \leq H_i^k \qquad \text{all } i,k \tag{29}$$

$$\sum_m h_{m,ij} = \sum_k \sum_w T_{ij}^{kw} \qquad \text{all } i,j \tag{30}$$

$$v_a = \sum_m \sum_i \sum_j \delta_{am,ij}\, h_{m,ij}$$

$$h_{m,ij} \geq 0 \tag{31}$$

$$T_{ij}^{kw} \geq 0 \tag{32}$$

The parameter μ is to be calibrated from observed data. We require that

$T_{ij}^{kw} \ell n\, T_{ij}^{kw} = 0$ at $T_{ij}^{kw} = 0$.

(P1) is the minimization of a strictly convex function subject to linear constraints. There is a unique solution and the Kuhn-Tucker conditions are necessary and sufficient for optimality. They are:

$$T_{ij}^{kw} = \exp\,(-1-v_j^w-\alpha_i^k)\,\exp\,[\mu(\tilde{\beta}_{ij}^{kw}-\theta u_{ij})] \tag{33}$$

$$h_{m,ij} > 0 \Rightarrow \sum_a s_a(v_a)\,\delta_{am,ij} = u_{ij} \tag{34}$$

$$h_{m,ij} = 0 \Rightarrow \sum_a s_a(v_a)\,\delta_{am,ij} \geq u_{ij} \tag{35}$$

$$\alpha_i^k > 0 \Rightarrow \sum_j \sum_w T_{ij}^{kw} = H_i^k \tag{36}$$

$$\sum_j \sum_w T_{ij}^{kw} < H_i^k \Rightarrow \alpha_i^k = 0 \tag{37}$$

where $U_{ij} = \mu\theta u_{ij}$, α_i^k and v_j^w are the dual variables associated with (30), (29) and (28) respectively.

If there were no congestion, or if we knew in advance the equilibrium travel times u_{ij}, (27) could be rewritten as:

$$\text{Max}\ -\frac{1}{\mu} \sum_{ijkw} T_{ij}^{kw} \ell n\, T_{ij}^{kw} + \sum_{wijk} (\hat{\beta}_{ij}^{kw} - \theta u_{ij})\, T_{ij}^{kw} \tag{27'}$$

(27') can be interpreted as the maximization of a "group willingness to pay" (Williams and Senior, 1978). On the other hand (27) has no such interpretation because of the presence of the quantity $\sum_a \int_0^{v_a} s_a(x)dx$. However (P1) ensures that the residential location that is generated by solving it is consistent with the residential location that would be generated by maximizing the "aggregate willingness to pay" (27') if the travel times were known in advance.

The quantities $\frac{1}{\mu}\alpha_i^k$ and $\frac{1}{\mu}\nu_j^w$ can be interpreted as location rents due to scarcity of the stock and as household taxes (if > 0) or subsidies (if < 0) arising from the fact that each household has to be located somewhere. The household taxes and subsidies $\frac{1}{\mu}\nu_j^w$ are transfer payments between households necessary because the prespecified "expected utility levels" are not in general consistent with the existing housing stock.

C Solution method

We will focus on the solution method for problem (P1) since we consider it preferable for practical applications to the model (H.S.C.). Recall that for a given value of μ, (P1) has a *unique* solution.

It is proposed first to calibrate μ on the basis of a survey and then to use the calibrated value of μ in (P1) and to solve the minimization problem (P1) directly to predict residential location in the projected year.

1) Calibration

Let us assume that we know the congestion function on each link of the road network and that an exhaustive survey allowed us to determine the allocations $\hat{T}_{ij}^{kw}$ and therefore the vacancies in the housing stock for each zone and each housing type. We observe an occupation of housing $\hat{H}_i^k = \sum_j \sum_w \hat{T}_{ij}^{kw}$. We know also the link flows $\hat{v}_a$, consistent with the observed $\hat{T}_{ij}^{kw}$, by using a traffic equilibrium algorithm. We attempt to calibrate μ in such a way that the solution of (P1) where (29) is replaced by

$$\sum_j \sum_w T_{ij}^{kw} = \hat{H}_i^k \qquad (29')$$

reproduces the observed values $\hat{T}_{ij}^{kw}$ and $\hat{v}_a$ when the calibrated value $\hat{\mu}$ is used in (27).

Let us introduce the following functions:

$$H \equiv \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw} \ln T_{ij}^{kw}$$

$$F \equiv \theta \sum_a \int_0^{v_a} s_a(x)dx - \sum_i \sum_j \sum_k \sum_w \hat{\beta}_{ij}^{kw} T_{ij}^{kw}$$

The objective function of (27) becomes: $H + \mu F$. By generalizing the results proved by Erlander, Nguyen and Stewart (1978) for the calibration of the combined distribution-assignment model it is possible to show that $F(\mu)$ and $H(\mu)$ are *monotone* functions of μ. (see Appendix 2). We have:

$$F(\mu) = \theta \sum_a \int_0^{v_a(\mu)} s_a(x)dx - \sum_i \sum_j \sum_k \sum_w \hat{\beta}_{ij}^{kw} T_{ij}^{kw}(\mu)$$

$$H(\mu) = \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw}(\mu) \ln T_{ij}^{kw}(\mu)$$

where $v_a(\mu)$ and $T_{ij}^{kw}(\mu)$ are the optimal solution to the minimization problem

$$\begin{cases} \text{Min} \quad H + \mu F \\ \text{s.t. (28), (29'), (30), (31), (32).} \end{cases}$$

Hence the calibration procedure for μ. Given $\hat{F}$ or $\hat{H}$ resulting from the survey, μ is a solution to one of the equations

$$F(\mu) = \hat{F} \qquad \text{or} \qquad H(\mu) = \hat{H}$$

It is possible to show that the corresponding flows $v_a(\mu)$ and allocations $T_{ij}^{kw}(\mu)$ are unique and should reproduce therefore the values observed in the survey. A dichotomic search procedure may be used to solve one of the two preceding equations for $\hat{\mu}$ (see Appendix 2).

2) Prediction

We now replace μ by its calibrated value. We solve directly problem (P1) with inequality constraints. (P1) consists in minimizing a convex function subject to linear constraints. This problem can be solved by a technique such as the Frank and Wolfe method [1] (1956). This method has proved successful in solving analogous problems in the past [2]. A typical iteration of this algorithm requires the solution of a linear program that determines the descent direction, and the solution of a one-dimensional minimization problem that determines the step size that achieves the best improvement in the objective function, given the descent direction. Consider an intermediate stage of the application of the algorithm when a feasible solution is known: $\bar{T}_{ij}^{kw}$ and path flows $\bar{h}_{m,ij}$. The descent direction is obtained by solving the problem:

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1. See Appendix 1 for a formulation of that method.

2. See Florian and Los (1977), Florian and Nguyen (1975, 1978), Florian, Nguyen and Ferland (1975).

$$\text{Min} \sum_i \sum_j \sum_k \sum_w \bar{\bar{c}}_{ij}^{kw} Y_{ij}^{kw} + \sum_i \sum_j \sum_m \bar{c}_{m,ij} Z_{m,ij} \tag{38}$$

$$\text{s.t.} \sum_i \sum_k Y_{ij}^{kw} = E_j^w \qquad \text{all } j,w \tag{39}$$

$$\sum_j \sum_w Y_{ij}^{kw} \leq H_i^k \qquad \text{all } i,k \tag{40}$$

$$\sum_m Z_{m,ij} = \sum_k \sum_w Y_{ij}^{kw} \qquad \text{all } i, j \tag{41}$$

$$Z_{m,ij} \geq 0, \quad Y_{ij}^{kw} \geq 0 \tag{42}$$

$$\text{where} \quad \bar{\bar{c}}_{ij}^{kw} = 1 + \ell n \, \bar{\bar{T}}_{ij}^{kw} - \mu \, \hat{\beta}_{ij}^{kw} \tag{43}$$

$\frac{1}{\mu\theta} \bar{c}_{m,ij}$ is the length of path m for OD pair (i,j) on the network when the flows are $\bar{h}_{m,ij}$. The linear program (38)-(42) simplifies by noting that $\frac{1}{\mu\theta} \bar{c}_{m,ij}$ may be replaced by $\frac{1}{\mu\theta} c_{ij}^*$ the length of the shortest path between every pair (i,j) on the network. In an optimal solution, if $\bar{c}_{m,ij} > c_{ij}^*$, then the corresponding $Z_{m,ij} = 0$, since otherwise the solution could be improved by diminishing a $Z_{m,ij}$ for which $\bar{c}_{m,ij} > c_{ij}^*$ and augmenting Z_{ij}^* (the flow on the shortest path). Thus the direction is obtained by the minimization of the objective function

$$\sum_i \sum_j \sum_k \sum_w \bar{\bar{c}}_{ij}^{kw} Y_{ij}^{kw} + \sum_i \sum_j c_{ij}^* \left(\sum_m Z_{m,ij} \right) \tag{44}$$

which simplifies to

$$\text{Min} \sum_i \sum_j \sum_k \sum_w \bar{\bar{c}}_{ij}^{kw} Y_{ij}^{kw} + \sum_i \sum_j c_{ij}^* \sum_k \sum_w Y_{ij}^{kw} \tag{45}$$

$$\text{or} \quad \text{Min} \sum_i \sum_j \sum_k \sum_w (\bar{\bar{c}}_{ij}^{kw} + c_{ij}^*) Y_{ij}^{kw} \tag{46}$$

The Linear Program (38)-(42) becomes:

$$\text{Min} \sum_i \sum_j \sum_k \sum_w \bar{c}_{ij}^{kw} Y_{ij}^{kw} \tag{47}$$

$$\text{s.t.} \sum_i \sum_j Y_{ij}^{kw} = E_j^w \qquad \text{all } j, w \tag{48}$$

$$\sum_j \sum_w Y_{ij}^{kw} \leq H_i^k \qquad \text{all } i, k \tag{49}$$

$$Y_{ij}^{kw} \geq 0 \tag{50}$$

$$\text{where } \bar{c}_{ij}^{kw} = \bar{\bar{c}}_{ij}^{kw} + c_{ij}^* \tag{51}$$

This is a transportation problem. This problem can be solved by an efficient code such as the code developed by Harris (1976). The solution elements are the components of the descent direction vector. We obtain $Z_{m,ij}$ by assigning $\sum_k \sum_w \bar{Y}_{ij}^{kw}$ for each (i,j) on the shortest route that was found while computing the c_{ij}^*. Let $\{\bar{Y}_{ij}, \bar{Z}_{m,ij}\}$ be the solution to the linear program defining the descent direction. The descent direction for the current iteration is:

$$(\bar{Y}_{ij}^{kw} - \bar{\bar{Y}}_{ij}^{kw}) \qquad \text{all } i,j,k,w \tag{52}$$

$$(\bar{Z}_{m,ij} - \bar{\bar{h}}_{m,ij}) \qquad \text{for all } m, i \text{ and } j \tag{53}$$

Note that $\bar{Z}_{m,ij} = 0$ for all m except for the shortest path m^* where

$$\bar{Z}_{m^*,ij} = \sum_k \sum_w \bar{Y}_{ij}^{kw} \tag{54}$$

The optimal step length λ^* to find the next feasible solution is given by the solution of the one dimensional minimization problem:

$$\text{Min} \sum_i \sum_j \sum_k \sum_w x_{ij}^{kw} \ell n\, x_{ij}^{kw} - \mu \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw} x_{ij}^{kw} + \mu\, \theta \sum_a F(\phi_a) \tag{55}$$

where

$$x_{ij}^{kw} = \bar{T}_{ij}^{kw} + \lambda(\bar{Y}_{ij}^{kw} - \bar{T}_{ij}^{kw}) \tag{56}$$

$$\phi_a = \bar{v}_a + \lambda(\bar{w}_a - \bar{v}_a) \tag{57}$$

$$\bar{w}_a = \sum_m \sum_i \sum_j \delta_{am,ij} \bar{Z}_{m,ij} \tag{58}$$

$$F(\phi_a) = \int_0^{\phi_a} s_a(x)dx \tag{59}$$

Note that the path flows $h_{m,ij}$ are redundant in the computation of the descent direction and of the step length λ^*. The explicit information on utilized paths between the OD pairs is not necessary. This property has been noted and used in several other cases.[1] The complete solution method to (P1) can be summarized as follows.

Step 1: Obtain an initial feasible solution $\{\bar{T}_{ij}^{kw}, \bar{v}_a\}$ to problem (P1).

Step 2: For each arc a, compute the current cost $\bar{c}_a = s_a(\bar{v}_a)$.

Step 3: For each O-D pair (i,j) determine the shortest path τ_{ij}, let $\frac{1}{\mu\theta} c_{ij}^*$ be the travel cost on τ_{ij}

Step 4: Compute: $\bar{c}_{ij}^{kw} = c_{ij}^* + 1 + \ell n\, \bar{T}_{ij}^{kw} - \mu\, \bar{\beta}_{ij}^{kw}$.

Step 5: Solve the transportation problem (47) to (50) to obtain $\bar{Y}_{ij}^{kw}$.

Step 6: Initialize $\bar{w}_a = 0$ for all arcs a. For all (i,j) set $\bar{w}_a = \bar{w}_a + \sum_k \sum_w \bar{Y}_{ij}^{kw}$ for $a \varepsilon \tau_{ij}$

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1. See Florian and Nguyen (1978) and Florian, Nguyen and Ferland (1975).

Step 7: Stopping criterion:

Compute

$$\Delta = \left| \sum_i \sum_j \sum_k \sum_w \bar{c}_{ij}^{kw} \left(\bar{Y}_{ij}^{kw} - \bar{T}_{ij}^{kw}\right) + \sum_a \bar{c}_a \left(\bar{w}_a - \bar{v}_a\right) \right|$$

If $\Delta \leq \varepsilon$ (a predetermined tolerance) stop; otherwise continue to the following step.

Step 8: Solve the one-dimensional problem (55) to determine the optimal step length λ^*. Revise the allocations and flows as follows:

$$\bar{T}_{ij}^{kw} = \bar{T}_{ij}^{kw} + \lambda^* \left(\bar{Y}_{ij}^{kw} - \bar{T}_{ij}^{kw}\right) \qquad \text{all } i,j,k,w$$

$$\bar{v}_a = \bar{v}_a + \lambda^* \left(\bar{w}_a - \bar{v}_a\right) \qquad \text{all } a$$

Return to step 2.

This solution procedure is very similar to the procedure proposed for the combined trip distribution-trip assignment by Ferland, Florian and Nguyen (1975).[1]

IV Models predicting residential location and modal choice

A A model of residential location, modal split and traffic assignment when all work-trips occur during peak-hours

Florian and Nguyen (1978) developed a combined trip distribution, modal split and trip assignment model. It is possible to synthesize their model with the residential location model (P1) above, thus allowing the simultaneous choice of residence and of mode of travel to work. The rationale for extending (P1) to include modal choice and make it endogenous to the model is that the "accessibility to work" advantage of a residence depends on the choice of mode: a predicted increase in con-

1. It is possible that some adaptation of Evans' algorithm (1976) for the combined trip distribution-trip assignment problem might be used instead of the Frank and Wolfe method. But given that we have inequality constraints here, in addition to equality constraints, this adaptation may not be trivial.

gestion on the road network due to an increased attraction of a residential zone can result either in a decreased demand for that zone or in a switch to another transportation mode. This interdependence between modal choice and housing choice has to be modelled if one is not to overstate the impact of automobile congestion on residential location. We will make the following assumptions.

There are two modes: the private automobile and the bus mode. As above the road network contains a set A of arcs a, each arc having its own congestion function $s_a(v_a)$. There is also a transit network[1] consisting of a set S of access arcs, transfer arcs and transit line segments. A transit line is composed of a number of segments. We associate a time c_s with each arc and line segment, $s \in S$, of the transit network as follows:

- a walking and waiting time with an access arc
- a walking and a waiting time with a transfer arc
- an in-vehicle time with a line segment.

If the link times c_s are not functions of the transit link volumes v_s, there is no congestion effect assumed on the transit network: travellers on the transit network choose the shortest path between an origin and a destination. Let u_{ij}^{tr} be the constant travel time from i to j on the transit network. The notation is the same as in previous sections with some modifications. Now β_{ij}^{kw}, the bid rent for a travel time equal to zero (and therefore $\hat{\beta}_{ij}^{kw}$), will be a function of mode, since it depends on the budget constraint and therefore on the *money* cost of travel, which differs with the choice of mode and appears in the budget constraint (See Section V

1. Here we follow closely Florian and Nguyen's (1978) notation and definitions.

below, eqs (95), (95'), (95")). We have then, $\hat{\beta}_{ij}^{kw,au}$ and $\hat{\beta}_{ij}^{kw,tr}$ respectively for car and transit.

$T_{ij}^{kw,au}$ is the number of heads of households that earn W, work in j, live in a type k house and use the car to travel to work.

$T_{ij}^{kw,tr}$ is the analogous quantity for heads of households that use transit.

$h_{m,ij}^{au}$ corresponds to the flow of private automobiles using path m between i and j, converted into person trips by the uniform car-occupancy factor.

The total flow of vehicles on link a (converted into persons) is:

$$v_a = \sum_i \sum_j \sum_m \delta_{am,ij} \, h_{m,ij}^{au} + v_a^{tr}$$

and v_a^{tr} is the vehicle equivalent of the number of buses that use link a (again converted into person-trips by using the uniform car-occupancy factor). v_a^{tr} may be computed by determining the number of private vehicles equivalent to a bus and by considering all the buses that use the link per time period. The assumptions made on the two modes imply that private automobiles do not have any influence on the speed or frequency of buses while buses do interfere with the traffic of automobiles and increase congestion. This assumes traffic management decisions giving priority to buses over cars on the road, for instance special bus lanes on which buses have absolute priority but open to cars in the absence of buses. (Such a system actually functions in Paris for instance)

We require $T_{ij}^{kw,au}$, $T_{ij}^{kw,tr}$ and $h_{m,ij}^{au}$ to be the solution to the following minimization problem:

$$(P2): \quad \text{Min} \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw,au} \ln T_{ij}^{kw,au} + \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw,tr} \ln T_{ij}^{kw,tr}$$

$$- \mu \{ \sum_w \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw,au} T_{ij}^{kw,au} + \sum_w \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw,tr} T_{ij}^{kw,tr}$$

$$- \theta [\sum_{a \in A} \int_0^{v_a} s_a(x)dx + \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw,tr} u_{ij}^{tr}] \} \qquad (60)$$

$$\text{s.t.} \quad \sum_j \sum_w (T_{ij}^{kw,au} + T_{ij}^{kw,tr}) \leq H_i^k \qquad \text{all } i,k \qquad (61)$$

$$\sum_i \sum_j (T_{ij}^{kw,au} + T_{ij}^{kw,tr}) = E_j^w \qquad \text{all } j,w \qquad (62)$$

$$\sum_k \sum_w T_{ij}^{kw,au} = \sum_m h_{m,ij}^{au} \qquad \text{all } i,j \qquad (63)$$

$$v_a = \sum_i \sum_j \sum_m \delta_{am,ij} h_{m,ij}^{au} + v_a^{tr}$$

$$T_{ij}^{kw,au} \geq 0 \qquad (64)$$

$$T_{ij}^{kw,tr} \geq 0 \qquad (65)$$

$$h_{m,ij}^{au} \geq 0 \qquad (66)$$

We let $T_{ij}^{kw,au} \ln T_{ij}^{kw,au} = 0$ at $T_{ij}^{kw,au} = 0$

and $T_{ij}^{kw,tr} \ln T_{ij}^{kw,tr} = 0$ at $T_{ij}^{kw,tr} = 0$

The objective function (60) is strictly convex in the variables $T_{ij}^{kw,au}$, $T_{ij}^{kw,tr}$ and $h_{m,ij}^{au}$. There is a unique solution and the Kuhn-Tucker conditions of (P2) are necessary and sufficient for optimality and are:

$$T_{ij}^{kw,au} = \exp\ (-1-\nu_j^w-\alpha_i^k)\ \exp\ [\mu(\hat{\beta}_{ij}^{kw,au}-\theta u_{ij}^{au})] \tag{67}$$

$$T_{ij}^{kw,tr} = \exp\ (-1-\nu_j^w-\alpha_i^k)\ \exp\ [\mu(\hat{\beta}_{ij}^{kw,tr}-\theta u_{ij}^{tr})] \tag{68}$$

$$h_{m,ij}^{au} > 0 \Rightarrow \sum_a s_a(v_a)\ \delta_{am,ij} = u_{ij}^{au} \tag{69}$$

$$h_{m,ij}^{au} = 0 \Rightarrow \sum_a s_a(v_a)\ \delta_{am,ij} \geq u_{ij}^{au} \tag{70}$$

$$\alpha_i^k > 0 \Rightarrow H_i^k = \sum_j \sum_w (T_{ij}^{kw,au} + T_{ij}^{kw,tr}) \tag{71}$$

$$H_i^k > \sum_j \sum_w (T_{ij}^{kw,au} + T_{ij}^{kw,tr}) \Rightarrow \alpha_i^k = 0 \tag{72}$$

The modal share for a quadruplet (i,j,k,w) is given by:

$$\frac{T_{ij}^{kw,au}}{T_{ij}^{kw,tr}+T_{ij}^{kw,au}} = \frac{\exp[\mu(\hat{\beta}_{ij}^{kw,au}-\theta u_{ij}^{au})]}{\exp[\mu(\hat{\beta}_{ij}^{kw,au}-\theta u_{ij}^{au})] + \exp\ [\mu(\hat{\beta}_{ij}^{kw,tr}-\theta u_{ij}^{tr})]} \tag{73}$$

From the definition of $\beta_{ij}^{kw,au}$ and $\beta_{ij}^{kw,tr}$ given in Section V (eqs 95, 95', 95") and from

$$\hat{\beta}_{ij}^{kw,tr} = \beta_{ij}^{kw,tr} - r_i^k$$

$$\hat{\beta}_{ij}^{kw,au} = \beta_{ij}^{kw,au} - r_i^k$$

we have

$$\hat{\beta}_{ij}^{kw,au} = K_i^{kw} - c_{ij}^{'au}$$

$$\hat{\beta}_{ij}^{kw,tr} = K_i^{kw} - c_{ij}^{'tr}$$

where K_i^{kw} is some constant dependent on i,k and w, and independent of mode choice. Therefore we have:

$$\frac{T_{ij}^{kw,au}}{T_{ij}^{kw,tr}+T_{ij}^{kw,au}} = \frac{\exp[-\mu(c_{ij}^{'au}+\theta u_{ij}^{au})]}{\exp[-\mu(c_{ij}^{'au}+\theta u_{ij}^{au})] + \exp[-\mu(c_{ij}^{'tr}+\theta u_{ij}^{tr})]} \qquad (74)$$

(74) represents a logit model with generalized costs " $c_{ij}^{'au} + \theta u_{ij}^{au}$ " and "$c_{ij}^{'tr} + \theta u_{ij}^{tr}$". ($c_{ij}^{'au}$ and $c_{ij}^{'tr}$ are the money costs of travel by car or transit). That expression is independent of w and k, and therefore (74) represents the modal share for all work-trips between i and j.

(P2) can be calibrated and solved by the same method as (P1), with

$$F \equiv \theta\Big[\sum_{a\epsilon A} \int_0^{v_a} s_a(x)dx + \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw,tr}\, u_{ij}^{tr}\Big]$$

$$- \Big[\sum_w \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw,au}\, T_{ij}^{kw,au} + \sum_w \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw,tr}\, T_{ij}^{kw,tr}\Big]$$

$$H \equiv \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw,au}\, \ell n\, T_{ij}^{kw,au} + \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw,tr}\, \ell n\, T_{ij}^{kw,tr}$$

The only difference is that the linear program that one would have to solve to find a descent direction in the Frank and Wolfe method would be a bimodal version of the Hitchcock problem. However, as shown by Florian and Nguyen (1978) this problem can be reduced to an equivalent single mode Hitchcock problem.

If the two alternative modes of travel were rail transit and the private car, one would have to extend to residential location the model developed by Los (1977) for modal choice, mode of access to rail, trip distribution and traffic assignment. One would obtain a model analogous to (P2) above. However the linear program one would have to solve to find a descent direction in the Frank and Wolfe algorithm would no longer be reducible to a Hitchcock problem.

B A model of residential location, modal split and traffic assignment when some work-trips occur during non-peak hours

A high proportion of work trips do occur outside peak hours. This fact has to be taken into account if one is not to overstate the impact of automobile congestion on residential location or modal split. We will assume that the proportion of the total number of work-trips taking place during peak hours is exogenous to the model. This assumption is reasonable in view of the fact that the hours of work during the day are largely determined institutionaly. We will use the following notation:

$T_{ij,\pi}^{kw,au}$ = total number of trips made by (j,w) heads of household living in an (i,k) residential bundle, using automobile during peak hours.

$T^{kw,au}_{ij,\pi}$ = total number of trips made by (j,w) heads of household living in an (i,k) residential bundle, using automobile outside peak hours.

$T^{kw,tr}_{ij,\pi}$ = total number of trips made by (j,w) heads of household living in an (i,k) residential bundle, using public transit during peak hours.

$T^{kw,tr}_{ij,\bar{\pi}}$ = total number of trips made by (j,w) heads of household living in an (i,k) residential bundle, using public transit outside peak hours.

$u^{tr}_{ij,\pi}$ = travel time from zone i to zone j on the transit network, during peak hours.

$u^{tr}_{ij,\bar{\pi}}$ = travel time from zone i to zone j on the transit network, outside peak hours.

$c^{au}_{ij,\bar{\pi}}$ = travel time from zone i to zone j on the automobile network, outside peak hours.

The quantities $u^{tr}_{ij,\pi}$, $u^{tr}_{ij,\bar{\pi}}$, $c^{au}_{ij,\bar{\pi}}$ are all constant. This implies that $c^{au}_{ij,\bar{\pi}}$ is obtained from the "free-flow" travel times on the links of the road network. The distinction between transit travel times $u^{tr}_{ij,\pi}$ and $u^{tr}_{ij,\bar{\pi}}$ allows for different qualities of service of the transit system (mainly frequency) during peak hours and non-peak hours. We require $T^{kw,au}_{ij,\pi}$, $T^{kw,au}_{ij,\bar{\pi}}$, $T^{kw,tr}_{ij,\pi}$ and $T^{kw,tr}_{ij,\bar{\pi}}$ to be the solution to the following minimization problem:

$$
\begin{aligned}
(P3): \quad \underset{}{\text{Min}} \; & \sum_i \sum_j \sum_k \sum_w T_{ij,\pi}^{kw,au} \, \ell n \, T_{ij,\pi}^{kw,au} + \sum_i \sum_j \sum_k \sum_w T_{ij,\bar{\pi}}^{kw,au} \, \ell n \, T_{ij,\bar{\pi}}^{kw,au} \\
& + \sum_i \sum_j \sum_k \sum_w T_{ij,\pi}^{kw,tr} \, \ell n \, T_{ij,\pi}^{kw,tr} + \sum_i \sum_j \sum_k \sum_w T_{ij,\bar{\pi}}^{kw,tr} \, \ell n \, T_{ij,\bar{\pi}}^{kw,tr} \\
& - \mu \Bigg\{ \sum_i \sum_j \sum_k \sum_w \hat{\beta}_{ij}^{kw,au} \left(T_{ij,\pi}^{kw,au} + T_{ij,\bar{\pi}}^{kw,au} \right) + \sum_i \sum_j \sum_k \sum_w \hat{\beta}_{ij}^{kw,tr} \left(T_{ij,\pi}^{kw} + T_{ij,\bar{\pi}}^{kw,tr} \right) \\
& \quad - \theta \Bigg[\sum_{a \in A} \int_0^{v_a} s_a(x) dx + \sum_i \sum_j \sum_k \sum_w T_{ij,\pi}^{kw,tr} \, u_{ij,\pi}^{tr} \\
& \quad + \sum_i \sum_j \sum_k \sum_w T_{ij,\bar{\pi}}^{kw,au} \, c_{ij,\bar{\pi}}^{au} + \sum_i \sum_j \sum_k \sum_w T_{ij,\bar{\pi}}^{kw,tr} \, u_{ij,\bar{\pi}}^{tr} \Bigg] \Bigg\} \qquad (75)
\end{aligned}
$$

$$
\text{s.t.} \qquad \sum_k \sum_w T_{ij,\pi}^{kw,au} = \sum_m h_{m,ij}^{au} \qquad \text{all } i,j \qquad (76)
$$

$$
\sum_i \sum_k \left(T_{ij,\pi}^{kw,au} + T_{ij,\pi}^{kw,tr} \right) = E_{j,\pi}^{w} \qquad (77)
$$

$$
\sum_i \sum_k \left(T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\bar{\pi}}^{kw,tr} \right) = E_{j,\bar{\pi}}^{w} \qquad (78)
$$

$$
\sum_j \sum_w \left[T_{ij,\pi}^{kw,au} + T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\pi}^{kw,tr} + T_{ij,\bar{\pi}}^{kw,tr} \right] \leq H_i^k \qquad (79)
$$

$$v_a = \sum_i \sum_j \sum_m \delta_{am,ij}\, h^{au}_{m,ij} + v^{tr}_a$$

$$T^{kw,au}_{ij,\pi} \geq 0,\ T^{kw,au}_{ij,\bar{\pi}} \geq 0,\ T^{kw,tr}_{ij,\pi} \geq 0,\ T^{kw,tr}_{ij,\bar{\pi}} \geq 0,\ h^{au}_{m,ij} \geq 0 \qquad (80)$$

Constraint (76) represents the conservation of flows constraint for the automobile trips during peak hours. Constraint (77) and constraint (78) allocate work-trips exogenously between peak hour and non-peak hour trips depending on the employment type or location. Constraint (79) is the housing supply contraint. Without this constraint (P3) would be decomposable into a peak hour prediction problem and a non-peak hour prediction problem. With this constraint, this decomposition is impossible. (75) is strictly convex in the variables $T^{kw,au}_{ij,\pi}$, $T^{kw,au}_{ij,\bar{\pi}}$, $T^{kw,tr}_{ij,\pi}$, $T^{kw,tr}_{ij,\bar{\pi}}$ and $h^{au}_{m,ij}$. There is a unique solution and the Kuhn-Tucker conditions, then necessary and sufficient for optimality are:

$$T^{kw,au}_{ij,\pi} = \exp[(-1-\nu^{w}_{j,\pi}-\alpha^{k}_{i})\exp[\mu(\hat{\beta}^{kw,au}_{ij} - \theta u^{au}_{ij,\pi})] \qquad (81)$$

$$T^{kw,au}_{ij,\bar{\pi}} = \exp(-1-\nu^{w}_{j,\bar{\pi}}-\alpha^{k}_{i})\exp[\mu(\hat{\beta}^{kw,au}_{ij} - \theta c^{au}_{ij,\bar{\pi}})] \qquad (82)$$

$$T^{kw,tr}_{ij,\pi} = \exp(-1-\nu^{w}_{j,\pi}-\alpha^{k}_{i})\exp[\mu(\hat{\beta}^{kw,tr}_{ij} - \theta u^{tr}_{ij,\pi})] \qquad (83)$$

$$T^{kr,tr}_{ij,\bar{\pi}} = \exp(-1-\nu^{w}_{j,\bar{\pi}}-\alpha^{k}_{i})\exp[\mu(\hat{\beta}^{kw,tr}_{ij} - \theta u^{tr}_{ij,\bar{\pi}})] \qquad (84)$$

$$h^{au}_{m,ij} > 0 \Rightarrow \sum_a s_a(v_a)\,\delta_{am,ij} = u^{au}_{ij,\pi} \qquad (85)$$

$$h^{au}_{m,ij} = 0 \Rightarrow \sum_a s_a(v_a)\,\delta_{am,ij} \geq u^{au}_{ij,\pi} \qquad (86)$$

$$\alpha^{k}_{i} > 0 \Rightarrow H^{k}_{i} = \sum_j \sum_w (T^{kw,au}_{ij,\pi} + T^{kw,au}_{ij\,\bar{\pi}} + T^{kw,tr}_{ij,\bar{\pi}} + T^{kw,tr}_{ij,\pi}) \qquad (87)$$

$$H^{k}_{i} > \sum_j \sum_w (T^{kw,au}_{ij,\pi} + T^{kw,au}_{ij,\bar{\pi}} + T^{kw,tr}_{ij,\bar{\pi}} + T^{kw,tr}_{ij,\pi}) \Rightarrow \alpha^{k}_{i} = 0 \qquad (88)$$

The modal share for a quadruplet (i,j,k,w) during peak hours is given by:

$$\frac{T^{kw,au}_{ij,\pi}}{T^{kw,tr}_{ij,\pi} + T^{kw,au}_{ij,\pi}} = \frac{\exp\left[-\mu(c'^{au}_{ij} + \theta u^{au}_{ij,\pi})\right]}{\exp\left[-\mu(c'^{au}_{ij} + \theta u^{au}_{ij,\pi})\right] + \exp\left[-\mu(c'^{tr}_{ij} + \theta u^{tr}_{ij,\pi})\right]} \quad (89)$$

During non-peak hours, the modal share is given by

$$\frac{T^{kw,au}_{ij,\bar{\pi}}}{T^{kw,tr}_{ij,\bar{\pi}} + T^{kw,au}_{ij,\bar{\pi}}} = \frac{\exp\left[-\mu(c'^{au}_{ij} + \theta c^{au}_{ij,\bar{\pi}})\right]}{\exp\left[-\mu(c'^{au}_{ij} + \theta c^{au}_{ij,\bar{\pi}})\right] + \exp\left[-\mu(c'^{tr}_{ij} + \theta u^{tr}_{ij,\bar{\pi}})\right]} \quad (90)$$

In both cases we have a logit model. However the total modal share (peak hours + non-peak hours) is not a logit model due to the effect of the dual variables $v^{tr}_{j,\pi}$ and $v^{w}_{j,\bar{\pi}}$ which may be distinct.

(P3) can be calibrated and solved in the same way as (P2). The linear program that one would have to solve to find a descent direction in the Frank and Wolfe method would now be a general linear program. This fact limits the level of disaggregation that can be used to the maximum size of the problems that can be solved by general purpose linear programming codes.

V Discussion and conclusion

A Bid rent estimation

1) Definition of bid rents

Let us assume that a household of type w has the following *linear* utility function[1].

$$U^{w} = M^{kw}_{ij} + \zeta^{w}.h^{k}_{i} - \theta c_{ij} \quad (91)$$

---

1. If the utility function is not linear it is linearized in a suitable way. This is discussed further below.

where M_{ij}^{kw} is the expenditure on non-locational goods if the household head works in j, lives in i in a house of type k and has income W;
h_i^k is a vector of housing characteristics for a type k house in zone i;
c_{ij} is the value of time spent going to work from i to j (It depends on the mode and possibly on automobile congestion).

The coefficients of the utility function ζ^w and θ have to be estimated empirically. The coefficient θ is assumed independent of w. This assumption is necessary for introducing congestion in the models. If in reality θ was dependent on the household type w we would have to compute a weighted average of the θ^w's. Let us assume that U^w, the level of utility for group w, is prespecified. For a (j,w) household choosing the (i,k) housing bundle we have the budget constraint:

$$W = b_{ij}^{kw} + M_{ij}^{kw} + c'_{ij} \tag{92}$$

where c'_{ij} is the annual money cost of travel between i and j (we could distinguish between c'^{au}_{ij} and c'^{tr}_{ij} if there were two modes, car and transit) and where b_{ij}^{kw}, the residual budget, is the bid rent of household (j,w). From (91) and (92) we obtain the bid rent as a function of the level of utility and of the value of time spent travelling to work.

$$b_{ij}^{kw} = W - U^w + \zeta^w.h_i^k - c'_{ij} - \theta c_{ij} \tag{93}$$

or
$$b_{ij}^{kw} = \beta_{ij}^{kw}(U^w) - \theta c_{ij} \tag{94}$$

where
$$\beta_{ij}^{kw}(U^w) = W - U^w + \zeta^w.h_i^k - c'_{ij} \tag{95}$$

When there are 2 modes, automobile and transit, we have:

$$\beta_{ij}^{kw,au} = W - U^w + \zeta^w.h_i^k - c'^{au}_{ij} \tag{95'}$$

$$\beta_{ij}^{kw,tr} = W - U^w + \zeta^w.h_i^k - c'^{tr}_{ij} \tag{95"}$$

The bid rents as given by (94) are used as *exogenous* data for the models of the previous sections.

2) Empirical estimation of bid rents

Two methods are *a priori* possible to estimate bid rents.

a) Direct estimation of the utility functions by household surveys

This is the method proposed by Harsman and Snickars (1975). Some methods are available to-day for the assessment of individual utility functions by means of interviews (See Keeney and Raiffa, 1976)

b) Use of econometric techniques

Anas (1975), Harris and Wheaton (1972) used such a technique. For instance one could estimate the coefficients of the following linear model on the basis of collected data:

$$M_{ij}^{kw} = U^{w} - \zeta^{w}.h_{i}^{k} + \theta\varepsilon_{ij} \qquad (96)$$

U^{w} would be the constant intercept of the linear regression model. On the basis of the coefficients thus estimated we could determine the bid rents as functions of the utility levels U^{w}. This method assumes that the market expenditures on housing or on non-locational goods, on which the multiple regression model is fitted, are the same as the preferred expenditures: market prices for housing are assumed equal to the bid rents at the "equilibrium" level of utility[1]. As soon as there is some control of prices (rent control for instance) the "observed" expenditures are not necessarily a reflection of preferences but may be a reflection of controls. If econometric techniques are to be used for estimating bid rents, they will have to take into account controlled prices and infer from observed expenditures, what they would have been in the absence of controls. If such an approach is at all feasible, it will require

1. This concept is discussed further below.

more research. In the meantime, the first approach may be the only feasible one in most cities (in particular in Canada).

3) The functional form of the utility functions.

It is possible (even probable) that household interviews or econometric techniques will lead to the conclusion that utility functions are non-linear (for instance loglinear). If this is the case the estimated utility functions have to be linearized first in order to obtain a bid rent which is a linear function of travel time. One way to do this is to linearize the utility function of a *representative* household in each household type around its *present* "situation" (i.e. consumption of housing, transport mode and other goods) therefore at the present level of utility (we can use a Taylor series expansion). This can be justified by the fact that "expectations" do not change abruptly in the short term or medium term. Otherwise we would have to make specific assumptions about the proper "typical situation" around which to linearize.

4) Endogenous determination of utility levels and bid rents

As shown by Harris and Wheaton (1972) and Wheaton (1974) the "equilibrium" bid rents (in Alonso's sense) are obtained at levels of utility such that household taxes and subsidies v_j^w are zero. With the models (P1 - P3) proposed in this paper the utility levels and therefore the bid rents could be adjusted in the manner proposed by Wheaton and Harris. We would have to add to their method a linearization mechanism to recompute the relevant value of θ at each adjustment of the utility levels U^w. The heuristic nature of the Harris-Wheaton mechanism would be compounded by this necessary recomputation of θ when utility functions are nonlinear. In fact, the assumption of utility-taking households may be reasonable in the medium

term for which the present models are proposed, even if we do not accept the assumption of an "open city".

B Limitations of the proposed models and possible extensions

The assumptions of exogenous supply of housing, of exogenous quantity and location of employment and of an exogenous quantity and distribution of households among types can be relaxed by interfacing one of the proposed models with a housing supply model, an employment location model and demographic or economic models at the level of the metropolitan area as a whole.

It would be easy to introduce in the models the choice of owning a car or not and another mode of transport such as a subway system. We might also make "accessibility" dependent on non-work trips such as shopping and recreation. Non-work trips could be made to contribute to congestion during peak hours (see Boyce and Southworth, 1978). The difference between "movers" and "non-movers" could be introduced by making the models "quasi-dynamic" (see Wilson, 1970). We could add parking supply constraints on a subset of zones of the city. This would have the result of limiting accessibility to work by car in those zones and would thereby influence modal choice in favor of the transit mode. Introducing all these modifications would increase the size of the linear programming subproblem in the Frank and Wolfe method. The increase in realism gained would have to be weighed against the increase in computational cost.

One serious limitation of the models proposed which will have to await more research before it can be relaxed is the assumption that there is

one worker per household who determines residential location. In many households at least two persons work and the choice of residence may be made with respect to both work locations. The resulting decision may be a "compromise". This aspect has to be studied empirically before it can be introduced into a model. However, the most serious limitation of the models proposed in this paper is the assumption of exogenous dwelling prices r_i^k. In most Canadian cities for instance some prices are controlled while others are free and adjust in a competitive fashion. For instance, in Montreal, as a first approximation, we can state that prices in the rental market are controlled while prices in the home ownership market are free to fluctuate according to the forces of supply and demand. These differences should be reflected in the use of the models for prediction. This can be done heuristically as follows. Model P1 (P2 or P3) outputs dual variables $[\frac{1}{\mu}\alpha_i^k]$ which can be interpreted as implicit location rents for dwellings of type k in zone i. For (i,k) housing choices for which prices are free we can reset the prices r_i^k at the values $r_i^k + \frac{1}{\mu}\alpha_i^k$ and reapply the model. We would iterate on these "free prices" until the associated dual variables converge towards zero. The "competitive prices" thus obtained would constitute *one* possible set of free prices. However this procedure is unsatisfactory for the following reasons:

- the final "free" prices might depend on the "initial" prices
- there may be a multiplicity of such prices and we have no reason to give more credence to some rather than to others.
- finally there are other aspects to the determination of house prices which have to be taken into account: the effect of the property tax for instance.

C Conclusion

This paper proposed for predicting residential location and transportation choices, models that can be solved by a convex programming technique such as the Frank and Wolfe method. A calibration method was proposed. This approach is computationaly feasible: the level of disaggregation chosen is only limited by the size of the problems that can be solved by the best codes for the Hitchcock problem (if all work-trips occur during peak hours and if there are no parking policy constraints) or the best linear programming codes (in the case of general models with peak-hour and non-peak-hour trips, parking policy constraints, etc...)

The proposed methodology is quite demanding in terms of data however. We must have a detailed description of household preferences for housing, a detailed description of the transportation system and of the housing stock. If no data were available about preferences for housing (bid rents) the models could still be used (by dropping the bid rent terms $\hat{\beta}_{ij}^{kw}$ from the models) but such an approach would explain residential household behavior only on the basis of accessibility characteristics of different locations. We would then obtain a rather sophisticated Lowry model, very sensitive to transportation policy measures.

The proposed models represent progress over previous residential location models because they integrate in a consistent mathematical programming framework residential location with previously proposed models of modal choice, trip distribution and traffic assignment. They take into account congestion on the road network and can predict imbalances between supply and demand in the housing stock by an explicit treatment of inequality

constraints in the prediction mechanism: there is an endogenous determination of the vacancies. Imbalances between supply and demand for housing are quite possible and actually observed. The proposed models should allow a very detailed study of the impact of transportation measures on housing choices (for instance the impact of a new subway line), or a detailed study of the impact of new housing construction on the utilization of the transportation networks. Because of the explicit treatment of income the impact of changes in income distribution on residential location and transport choices could be predicted. The impact of modifications of controlled rents on the utilization of the housing stock could be easily predicted and the implicit location rents produced by the dual variables associated with the housing supply constraints could provide valuable information to urban planners by showing which constraints are binding and which are not. Finally, the unicity of the locational equilibrium obtained under the exogenous conditions and the fact that the algorithm proposed is proved to converge towards this unique equilibrium (Frank and Wolfe, Zangwill) is extremely useful for policy sensitivity analysis.

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APPENDIX 1

Proof that H and F are monotone functions of μ and that the calibration procedure proposed for model (P1) leads to unique values of T_{ij}^{kw} and v_a which reproduce the survey data.[1]

The objective function of (P1) is

$$H + \mu F \quad ,$$

with
$$F \equiv \theta \sum_{a \in A} \int_0^{v_a} s_a(x)dx - \sum_i \sum_j \sum_k \sum_w \hat{\beta}_{ij}^{kw} T_{ij}^{kw}$$

$$H \equiv \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw} \ln T_{ij}^{kw}$$

For a given value of μ, $H(\mu) + \mu F(\mu)$, $v_a(\mu)$ and $T_{ij}^{kw}(\mu)$ are the value of the objective function and the solution to the minimization problem

$$\text{Min} \quad H + \mu F$$

s.t. (28), (29'), (30), (31), (32)

1) For $0 < \mu < \infty$, $F(\mu)$ is monotonically decreasing and $H(\mu)$ is monotonically increasing with μ.

Proof: For given positive μ_1 and μ_2, we have:

(A1) $\mu_2 [F(\mu_2) - F(\mu_1)] \leq H(\mu_1) - H(\mu_2) \leq \mu_1 [F(\mu_2) - F(\mu_1)]$

(A2) and $(\mu_2 - \mu_1) [F(\mu_2) - F(\mu_1)] \leq 0$

Hence:

(A3) $\mu_2 > \mu_1 \Rightarrow \begin{matrix} F(\mu_2) - F(\mu_1) \leq 0 \\ H(\mu_2) - H(\mu_1) \geq 0 \end{matrix}$

---

1. This proof restates with slight modifications the proof by Erlander, Nguyen and Stewart (1978) for the case of the combined distribution assignment.

2) If either F or H is constant on an interval $[\mu_1,\mu_2]$, $0<\mu_1<\mu_2$, all of the following are constant for $\mu\varepsilon[\mu_1,\mu_2]$:

$$F,\ H,\ T_{ij}^{kw},\ v_a.$$

Proof:

a) Suppose that $F(\mu_1) = F(\mu_2)$

(A1) implies $H(\mu_1) = H(\mu_2)$

It is a strictly convex function of T_{ij}^{kw} and therefore if H is constant on $[\mu_1,\mu_2]$ the variables T_{ij}^{kw} are. F is a strictly convex function of the variables v_a (but not of the variables v_a *and* T_{ij}^{kw}). Since F is constant on $[\mu_1,\mu_2]$ and given that the T_{ij}^{kw} are constant when H is, the v_a are uniquely defined also.

b) Suppose that $H(\mu_1) = H(\mu_2)$

(A1) implies $F(\mu_1) = F(\mu_2)$

H being constant on an interval $[\mu_1,\mu_2]$ and strictly convex in the T_{ij}^{kw} , the T_{ij}^{kw} are uniquely defined. F being constant on $[\mu_1,\mu_2]$ and a strictly convex function in the v_a , the v_a are uniquely defined when F and the T_{ij}^{kw} are constant.

APPENDIX 2

The Frank and Wolfe algorithm (1956)

Consider the problem of minimizing a convex function subject to linear constraints:

$$\text{Min } Z(x) \text{ , subject to } A X \leq b \text{ , } X \geq 0.$$

The detailed steps of the Frank Wolfe algorithm for solving this problem are:

Step 1 Given a feasible solution X^1 , set $\ell = 1$

Step 2 Determine Y^ℓ that minimizes

$\nabla Z(X^\ell)\, Y$, subject to $A Y \leq b$, $Y \geq 0$.

Step 3 Set the descent direction $d^\ell = Y^\ell - X^\ell$.

If $| \nabla Z(X^\ell)\, d^\ell | \leq \varepsilon$, terminate, where ε is a suitable convergence parameter. (X^ℓ is the optimal solution).

Step 4 Find the optimal step length λ^ℓ that minimizes

$Z(X^\ell + \lambda\, d^\ell)$ for $0 \leq \lambda \leq 1$.

Step 5 Revise the current solution

$X^{\ell+1} = X^\ell + \lambda^\ell\, d^\ell$;

set $\ell = \ell + 1$ and return to Step 2.